HERO OF ALEXANDRIA'S NUMERICAL TREATMENT OF DIVISION IN EXTREME AND MEAN RATIO AND ITS IMPLICATIONS

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"*Mirabilis itaque est potentia linee secundum proportionem habentem medium duoque extrema divise. Cui cum plurima* [sic] *pholosophantium* [sic] *admiratione digna conveniant hoc principium vel praecipuum ex superiorum principiorum invariabili procedit natura ut tam diversa solida tum magnitudine tum basium numero tum etiam figura irrationali quadam simphonia rationabiliter conciliet.*" Campanus of Novara (13th century) on Book XIV, prop. 10 of Euclid's *Elements*.[1]

Perhaps the key assumption in Duckworth's study of the *Aeneid*[2] is that Virgil was not only aware of the geometrical concept of division in extreme and mean ratio[3] and considered it a means of introducing ratios in his poems, but that he was also aware of approximations to this ratio in the form of ratios of successive terms in the Fibonacci sequence.[4] The main purpose of this note is to comment on the possibility of a knowledge of the Fibonacci ratios in the classical period.

Although Dalzell, Clarke, and Bews[5] have argued *ex silentio* that there is no evidence that the Fibonacci sequence was known in classical times, we can in fact be more precise, for we have an explicit indication of the treatment of a practical problem involving division in extreme and mean ratio in the period just following Virgil. This indication appears in the *Metrica* of Hero of Alexandria, who was active circa A.D. 62.[6]

In *Metrica* 1.17–18 Hero wishes to find the area of a pentagon whose

[1]Campanus of Novara, in *Elementa geometriae Euclidis, latine, cum annotationibus Campani* (Venice 1482; cited from Paris, Bibliothèque Nationale Rés. g. V. 33).

[2]G. Duckworth, *Structural Patterns and Proportions in Vergil's "Aeneid"* (Ann Arbor 1962). See also E. Brown, *Numeri Vergiliani: Studies in 'Eclogues' and 'Georgics'* (Brussels 1963 [*Collection Latomus* 63]).

[3]A line is said to be cut in extreme and mean ratio if the larger segment is to the smaller as the whole line is to the larger segment (Euclid 6, definition 3).

[4]In modern terminology the number or ratio determined by "division in extreme and mean ratio" would have the value (usually called the golden number, section, cut, ratio, etc) 1.618 .... The approximating Fibonacci ratios are 5/3, 8/5, 13/8 ....

[5]A. Dalzell, review of Duckworth, *Phoenix* 17 (1963) 314–316; M. L. Clarke, "Vergil and the Golden Section," *CR* N.S. 14 (1964) 43–45; J. Bews, "*Aeneid* I and .618?", *Phoenix* 24 (1970) 130–143.

[6]The relevant portion of the *Metrica* is given in H. Schöne (ed.), *Heronis Alexandrini Opera quae supersunt omnia* 3 (Leipzig 1903) 50–53. The text, together with a reproduction of the manuscript, scholia, and commentaries appears in E. Bruins (ed.), *Codex Constantinopolitanus* (Leiden 1964) 1.149–151, 2.100–101, 3.224–229, For the date of Hero as well as a discussion of his work see M. Mahoney and A. Drachmann in *Dictionary of Scientific Biography* (New York 1970) 6.310–314, 314–335.

Phoenix, Vol. 35 (1981) 2.

side is 10 units. Let us consider Hero's procedure with the aid of diagrams and modern terminology and notation.

To find the area of the pentagon ABCDE (figure 1), whose centre is Z, one first finds the area of the triangle CZD and then multiplies by five. The area of the triangle in turn is one half of the product of the base CD and the altitude ZH. Since the base CD is given to be 10 units one need only find the altitude ZH.

Note that the central angle of the pentagon is 72° so that in the right triangle CZH the vertex angle CZH is 36°. This is where Hero's ingenuity becomes clear. He remarks in a lemma that not only is 36° one half of the central angle of the pentagon but it is also (figure 2) the angle FGJ between the side FG of a pentagon and its diagonal JG. It is at this point that division in extreme and mean ratio appears, for Euclid 13.8 states that the diagonal and side of a pentagon are in the same ratio as a line divided in extreme and mean ratio.

This is the ideal place for Hero to have used the ratio 8:5 of two consecutive Fibonacci numbers as an approximation to the desired ratio.

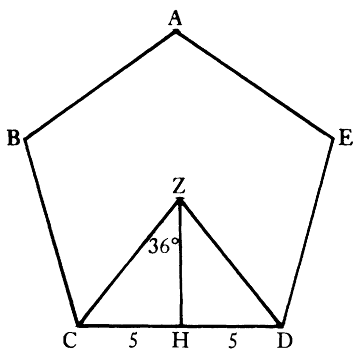

FIGURE 1.

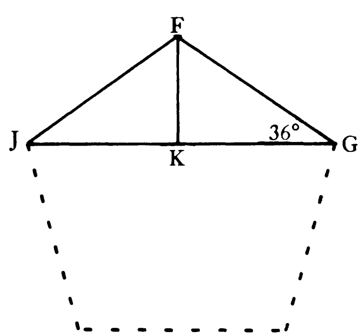

FIGURE 2.

One would simply say that since JG is to FG approximately as 8 is to 5, FG and KG, and therefore CZ and ZH in the original pentagon, are in the approximate ratio 5 to 4.

Hero, however, does not proceed in this simple fashion. Having, as stated above, used Euclid 13.8 to show JG and FG are in the same ratio as a line divided in extreme and mean ratio, he now invokes Euclid 13.1. This result says that since KG is half of JG we have the relationship

$$(FG + KG)^2 = 5(KG)^2$$

and consequently in the original pentagon

$$(1)\quad (CZ + ZH)^2 = 5(ZH)^2.$$

Now follows an approximation argument: if we take, for ratio purposes, $ZH = 4$, then $5(ZH)^2 = 80$. But this is not a perfect square so 80 is approximated by 81 and (1) gives $CZ + ZH = 9$ or $CZ = 5$. This shows once again, as in our simple argument above, that, CZ and ZH are in the approximate ratio 5 to 4.[7]

What must be emphasized is that we have here neither a mathematician interested in proving theoretical niceties nor some simple compiler of empirical formulae, but a good mathematician who dealt with both theoretical and applied geometry and who also gives every indication of being familiar with a large body of knowledge, including approximation techniques, dating back to the Babylonian period.[8]

[7]The rest of the computation, which is irrelevant to our argument, can be summarized as follows: Since CZ and ZH are the hypotenuse and leg of a right triangle and are in the approximate ratio 5 to 4, the Pythagorean theorem shows that ZH and the half base CH are in the approximate ratio 4 to 3. Consequently the ratio of the area of triangle CZD to the square of the side CD is approximately as 1 to 3 and in turn the ratio for the pentagon and the square of the side CD is as 5 to 3. We also note that in Hero's *Geometrica* both the ratios 5 to 3 and 12 to 7 are given; J. Heiberg (ed.), *Heronis Alexandrini Opera quae supersunt omnia* 4 (Leipzig 1911) 394, 382; also Bruins (above, n.6) 2.33, 118, 3.51, 172. No indication is given of the derivation of the ratio 12 to 7, but it does not seem to have any connection with either the Fibonacci ratios or Hero's remark at the end of *Metrica* 1.18, which reads: "And if we should take another square being nearer to the fivefold of another square, we shall find the area more precisely." Bruins (3.63) points out that the next good approximation (i.e., "convergent," though not necessarily calculated using continued fractions or the Euclidean algorithm) of the ratio root 5 to 1, after 9 to 4, would be 38 to 17 and concludes that this is the improvement alluded to by Hero. Bruins does not point out, however, that the resulting approximating ratio for the pentagon is no longer a ratio of whole numbers. In a private communication D. H. Fowler observes that only the particular choice of the Fibonacci ratio 8 to 5 will result in a ratio of whole numbers. He also notes, in passing, that the ratios 5 to 3 and 12 to 7 given by Hero happen to be "convergents" of the continued fraction expansion of the ratio of the area of the pentagon to the square of the side. For further details of a reconstruction of a continued fraction ("anthyphairetic") theory of ratio, see the articles by Fowler: "Ratio in Early Greek Mathematics." *Bull. Amer. Math. Soc.* 1 (1979) 807–846, and "Book II of Euclid's 'Elements' and a pre-Eudoxian theory of ratio," *Arch. Hist. Exact Sciences* 22 (1980) 5–36.

[8]According to Arab sources, Hero is responsible for the geometrical method of dividing a line in extreme and mean ratio that is now widely used; see M. Curtze (ed.), *Anaritii*

The minimum conclusion that we wish to draw is that the approximation of division in extreme and mean ratio was not a standard procedure, even in the most competent mathematical circles, during the classical period.[9]

Since our examination of the question of Fibonacci approximations of division in extreme and mean ratio in classical times was motivated by Duckworth's book it is appropriate to address ourselves briefly to the question of explaining Duckworth's numerical results.

Starting from a remark of A. Dalzell[10] one of the authors has made a detailed study[11] of the implications of the use of the ratio M/(m + M), which Duckworth employs, rather than the ratio m/M. The conclusion

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in decem libros Elementorum Euclidis commentarii ex interpretatione Gheradi Cremonenesis. On Hero as a recipient of ancient knowledge see O. Neugebauer *The Exact Sciences in Antiquity* (Providence, 1957), 46, 146, as well as the various comments by Bruins.

[9]Mention must be made here of a recent article by W. Knorr, "Archimedes and the Measurement of the Circle: A New Interpretation," *Arch. Hist. Exact Sciences* 15 (1975/76) 115–140. In brief Hero (Bruins [above, n.6] 2.105, 3.244) gave some presumably corrupt approximations for the ratio of the circumference of a circle to its diameter that "... Archimedes showed in 'On Blocks and Cylinders' ...." Knorr proposes an emendation that can be explained by assuming that Archimedes employed very accurate Fibonacci approximations to extreme and mean ratio. Whether the thesis of Knorr is correct or not, one may still argue that Hero does not seem to have had the idea of using Fibonacci approximations. It is possible that only the final numbers and not the method survived until Hero's time or that Archimedes had not given the details in the first place. Knorr (136) suggests that Hero did not understand some of the more subtle theoretical aspects of Archimedes' work. There has also been some discussion (cf. S. Heller, "Die Entdeckung der stetigen Teilung durch die Pythagoreer," *AbhBerl, Klasse Mathematik, Physik, Technik,* 1968. 6, 5–28; W. Knorr, *The Evolution of the Euclidean Elements* [Dordrecht 1975] 34) of scholium 73 to Euclid 2.11 (J. Heiberg ed., *Opera Omnia* 5 [Leipzig 1888] 249) and the possibility that it reflects a knowledge in ancient times of methods of generating Fibonacci approximations to division in extreme and mean ratio. Here we shall simply state that in our opinion Heller's interpretation is based on an incorrect translation of the Greek and that Knorr incorrectly assumes that the scholiast takes the segment HB to be equal to 3 units in the first place. In our interpretation HB was taken to be 3 1/8 and only later was the product term due to the 1/8 neglected. From this it would appear that this scholium of unknown date is more likely to represent crude approximations and numerical illustrations rather than an advanced knowledge of Fibonacci approximations. We know of only one numerical calculation in Greek mathematics that explicitly deals with division in extreme and mean ratio. This appears in Ptolemy's *Amalgest* (2nd century A.D.). In I10 the chord of 36°, i.e., the side of a regular decagon, whose length is just the larger segment of the radius when it is divided in extreme and mean ratio, is obtained. Essentially the method consists of computing 1/2 (root 5 − 1). It would appear that in the earlier table of Hipparchus (ca 150 B.C.) the chord of 36° was not computed; see G. Toomer, "The Chord Table of Hipparchus and the Early History of Greek Trigonometry," *Centaurus* 18 (1974) 6–28.

[10]Above, n.5.

[11]R. Fischler, "How to Find the 'Golden Number' Without Really Trying," to appear in the *Fibonacci Quarterly* (Dec. 1981).

was that the use of the former is not statistically valid in that it compresses and distorts the given data. Calculations showed that if the given data were in fact completely random ("uniformly distributed") then the use of the ratio M/(m + M) would distort the deviation of means, the standard deviation and probability statements by a factor of about 2.6. Furthermore it was suggested that indeed the true situation in most of the related "golden number" literature was that a certain range of values was inherent in the author's style, the author naturally obtaining values scattered randomly over this interval with a precise mathematical formula being the furthest thing from his mind. An analysis, using the ratio m/M, has now been made with Duckworth's data and indicates that random scattering is indeed the case with Virgil.[12]

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[12]If we compute the values m/M for entries 1–100 in Duckworth's table I we find that the range of these values is from 4/7 = .571 to 2/3 = .667. If this range is split up into five equal parts, then the five subintervals contain 10, 25, 33, 15, 17 values respectively. The Fibonacci ratios 3/5, 5/8, 13/21 appear 15, 16, 2 times respectively; in other words 2/3 of the ratios are *not* Fibonacci ratios, with the end values alone appearing 16 times. If we check the 16 values in table VI we see that not a single Fibonacci ratio appears. Further computational details are available from the second author. For the history of another attempt to read the "golden number" into classical literature see R. Fischler, "What did Herodotus Really Say? or How to Build (a Theory of) the Great Pyramid," *Environment and Planning B* 6 (1979) 89–93.